

Detecting sound hard cracks in isotropic inhomogeneities

Lorenzo Audibert^{1,2}, Lucas Chesnel², Houssem Haddar², and Kevish Napal²

Abstract We consider the problem of detecting the presence of sound-hard cracks in a non homogeneous reference medium from the measurement of multi-static far field data. First, we provide a factorization of the far field operator in order to implement the Generalized Linear Sampling Method (GLSM). The justification of the analysis is also based on the study of a special interior transmission problem. This technique allows us to recover the support of the inhomogeneity of the medium but fails to locate cracks. In a second step, we consider a medium with a multiply connected inhomogeneity assuming that we know the far field data at one given frequency both before and after the appearance of cracks. Using the Differential Linear Sampling Method (DLSM), we explain how to identify the component(s) of the inhomogeneity where cracks have emerged. The theoretical justification of the procedure relies on the comparison of the solutions of the corresponding interior transmission problems without and with cracks. Finally we illustrate the GLSM and the DLSM providing numerical results in 2D. In particular, we show that our method is reliable for different scenarios simulating the appearance of cracks between two measurements campaigns.

Key words: inverse scattering, sampling methods, sound-hard cracks.

1 Introduction

This work is a contribution to sampling methods in inverse scattering theory when the issue is to determine the shape of an unknown inclusion from fixed frequency multi-static data. More precisely we extend the Generalized Linear Sampling Method (GLSM) and the Differential Linear Sampling Method (DLSM) [2, 3] to inhomogeneities.

¹ Department PRISME, EDF R&D, 6 quai Watier, 78401, Chatou CEDEX, France.

² INRIA/Centre de mathématiques appliquées, École Polytechnique, Université Paris-Saclay, Route de Saclay, 91128 Palaiseau, France. e-mail: lorenzo.audibert@edf.fr, lucas.chesnel@inria.fr, houssem.haddar@inria.fr, kevish.napal@cmap.polytechnique.fr

geneous media containing sound-hard cracks. GLSM provides an exact characterization of the target shape from the far field operator, and its implementation mainly requires two complementary factorizations of the far field operator, one used in the Linear Sampling Method (LSM) and another used in the Factorization Method (FM). From the measurements for both the damaged background and the initial background, it is possible to detect the defect thanks to the DLSM. This method consists in combining a result of comparison of two interior transmission problems associated with each background and the results of the GLSM.

The purpose of this article is to establish a similar factorization for a medium containing sound-hard cracks and to provide the theoretical results needed in the justification of the DLSM, the method we use to identify emergence of defects in an unknown background. For references of works dealing with qualitative methods to detect cracks, we mention, among others, [5, 6, 11, 4].

2 The forward scattering problem

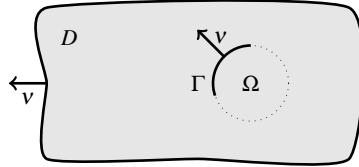


Fig. 1: Example of setting in \mathbb{R}^2 .

We consider an isotropic medium embedded in \mathbb{R}^d , $d = 2$ or 3 , containing sound-hard cracks. Following [6], a crack Γ is defined as a portion of a smooth nonintersecting curve ($d = 2$) or surface ($d = 3$) that encloses a domain Ω , such that its boundary $\partial\Omega$ is smooth. We assume that Γ is an open set with respect to the induced topology on $\partial\Omega$. The normal vector v on Γ is defined as the outward normal vector to Ω (see Fig. 1). To define traces and normal derivatives of functions on Γ , we use the following notation for all $x \in \Gamma$:

$$f^\pm(x) = \lim_{h \rightarrow 0^+} f(x \pm h v(x)) \quad \text{and} \quad \partial_v^\pm f(x) = \lim_{h \rightarrow 0^+} v(x) \cdot \nabla f(x \pm h v(x)).$$

We shall also work with the jump functions

$$[f] := f^+ - f^- \quad \text{and} \quad \left[\frac{\partial f}{\partial v} \right] := \partial_v^+ f - \partial_v^- f.$$

We assume that the propagation of waves in time harmonic regime in the reference medium is governed by the Helmholtz equation $\Delta u + k^2 u = 0$ in \mathbb{R}^d where Δ stands for the Laplace operator of \mathbb{R}^d and where k is the wave number. We assume that the cracks are embedded in a local perturbation of the reference medium. To model this perturbation, we introduce $n \in L^\infty(\mathbb{R}^d)$ a complex valued coefficient (the refractive index of the medium) such that $n = 1$ in $\mathbb{R}^d \setminus \overline{D}$ and $n \neq 1$ in D . Here $D \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary ∂D such that $\mathbb{R}^d \setminus D$ is connected. We assume that $\Im(n) \geq 0$ in \mathbb{R}^d and that $\Gamma \subset D$. The scattering of the incident plane wave $u_i(\theta, \cdot) := e^{ik\theta \cdot x}$ of direction of propagation $\theta \in \mathbb{S}^{d-1}$ by the medium is described by the problem

$$\begin{cases} \text{Find } u = u_i + u_s \text{ such that} \\ \Delta u + k^2 n u = 0 \text{ in } \mathbb{R}^d \setminus \Gamma \\ \partial_\nu^\pm u = 0 \text{ on } \Gamma \\ \lim_{r \rightarrow +\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u_s}{\partial r} - iku_s \right) = 0, \end{cases} \quad (1)$$

with $u_i = u_i(\theta, \cdot)$. The last line of (1), where $r = |x|$, is the Sommerfeld radiation condition which selects the outgoing scattered field and which is assumed to hold uniformly with respect to $\hat{x} = x/|x| \in \mathbb{S}^{d-1}$. For all $k > 0$, Problem (1) has a unique solution u belonging to $H^1(\mathcal{O} \setminus \Gamma)$ for all bounded domain $\mathcal{O} \subset \mathbb{R}^d$. The scattered field $u_s(\theta, \cdot)$ has the expansion

$$u_s(\theta, x) = \eta_d e^{ikr} r^{-\frac{d-1}{2}} \left(u_s^\infty(\theta, \hat{x}) + O(1/r) \right), \quad (2)$$

as $r \rightarrow +\infty$, uniformly in $\hat{x} = x/|x| \in \mathbb{S}^{d-1}$. In (2) the constant η_d is given by $\eta_d = e^{i\frac{\pi}{4}}/\sqrt{8\pi k}$ for $d = 2$ and by $= 1/(4\pi)$ for $d = 3$. The function $u_s^\infty(\theta, \cdot) : \mathbb{S}^{d-1} \rightarrow \mathbb{C}$, is called the far field pattern associated with $u_i(\theta, \cdot)$. From the far field pattern, we can define the far field operator $F : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1})$ such that

$$(Fg)(\hat{x}) = \int_{\mathbb{S}^{d-1}} g(\theta) u_s^\infty(\theta, \hat{x}) d\sigma(\theta). \quad (3)$$

By linearity, the function Fg corresponds to the far field pattern of the scattered field in (1) with

$$u_i = v_g := \int_{\mathbb{S}^{d-1}} g(\theta) e^{ik\theta \cdot x} d\sigma(\theta) \quad (\text{Herglotz wave function}). \quad (4)$$

3 Factorization of the far field operator

In this section we explain how to factorize the far field operator F defined in (3). From the Green representation theorem, computing the asymptotic behaviour of the Green's function as $r \rightarrow +\infty$ gives

$$u_s^\infty(\hat{x}) = \left(k^2 \int_D (n(y) - 1) u(y) e^{-ik\hat{x}y} dy + \int_\Gamma [u(y)] \partial_v^+ e^{-ik\hat{x}y} ds(y) \right) \quad (5)$$

for the far field pattern of u_s in (2). A first step towards the factorization of F is to define the Herglotz operator $H : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(D) \times L^2(\Gamma)$ such that

$$Hg = (v_g|_D, \partial_v^+ v_g|_\Gamma). \quad (6)$$

We give in Proposition 1 below a characterization of the closure of the range of H . Set

$$\mathcal{H} = \{v \in L^2(D) \mid \Delta v + k^2 v = 0 \text{ in } D\}. \quad (7)$$

and define the map $\Psi : \mathcal{H} \rightarrow L^2(D) \times L^2(\Gamma)$ such that

$$\Psi v = (v|_D, \partial_v^+ v|_\Gamma). \quad (8)$$

Proposition 1. *The operator $H : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(D) \times L^2(\Gamma)$ defined in (6) is injective and $\overline{R(H)} = \Psi(\mathcal{H})$.*

Proof. The proof of the injectivity of H follows a classical argument based on the Jacobi Anger expansion (apply [7, Lemma 2.1]). To establish the second part of the claim, first we note that v_g (defined in (4)) belongs to \mathcal{H} so that $R(H) \subset \Psi(\mathcal{H})$. On the other hand, classical results of interior regularity ensure that is some constant $C > 0$ such that $\|\partial_v v\|_{L^2(\Gamma)} \leq C\|v\|_{L^2(D)}$ for all $v \in \mathcal{H}$. This in addition to $\|\Psi v\|_{L^2(D) \times L^2(\Gamma)} \geq \|v\|_{L^2(D)}$ allows one to show that $\Psi(\mathcal{H})$ is a closed subspace of $L^2(D) \times L^2(\Gamma)$. The regularity result implies that $\Psi : (\mathcal{H}, \|\cdot\|_{L^2(D)}) \rightarrow L^2(D) \times L^2(\Gamma)$ is continuous. Since the set of Herglotz wave functions is dense in $(\mathcal{H}, \|\cdot\|_{L^2(D)})$, we deduce that $\overline{R(H)} = \Psi(\mathcal{H})$. \square

Next we define the operator $G : \overline{R(H)} \rightarrow L^2(\mathbb{S}^{d-1})$ such that

$$G(v, \partial_v^+ v) = u_s^\infty, \quad (9)$$

where u_s^∞ is the far field pattern of u_s , the outgoing scattered field which satisfies

$$\begin{cases} \Delta u_s + k^2 n u_s = k^2(1-n)v & \text{in } \mathbb{R}^d \setminus \Gamma \\ \partial_v^\pm u_s = -\partial_v^\pm v & \text{on } \Gamma. \end{cases} \quad (10)$$

Note that if $(v, \partial_v^+ v) \in \overline{R(H)}$ then interior regularity implies $\partial_v^+ v = \partial_v^- v$ on Γ . We also define the map $T : L^2(D) \times L^2(\Gamma) \rightarrow L^2(D) \times L^2(\Gamma)$ such that

$$T(v, \partial_v^+ v) = (k^2(n-1)(v+u_s), [v+u_s]). \quad (11)$$

Clearly we have $F = GH$. And one can check using (5) that $G = H^*T$ so that F admits the factorisation

$$F = H^*TH. \quad (12)$$

The justification of the techniques we propose below to recover the cracks will depend on the properties of the operators G, T . And the latter are related to the solvability of the so-called interior transmission problem which in our situation states as follows: given $f \in H^{3/2}(\partial D), g \in H^{1/2}(\partial D)$

$$\left| \begin{array}{l} \text{Find } (u, v) \in L^2(D) \times L^2(D) \text{ such that} \\ w := u - v \in \{\varphi \in H^1(D \setminus \Gamma) \mid \Delta \varphi \in L^2(D \setminus \Gamma)\} \\ \Delta u + k^2 n u = 0 \text{ in } D \setminus \Gamma \quad u - v = f \text{ on } \partial D \\ \Delta v + k^2 v = 0 \text{ in } D \quad \partial_v u - \partial_v v = g \text{ on } \partial D \\ \partial_v^\pm u = 0 \text{ on } \Gamma. \end{array} \right. \quad (13)$$

We shall say that $k > 0$ is a transmission eigenvalue if (13) with $f = g = 0$ admits a non zero solution. One can show for example that if the coefficient n is real and satisfies $1 < n_* < n < n^*$ for some constants n_* , n^* , then the set of transmission eigenvalues is discrete without accumulation point and that Problem (13) is uniquely solvable if and only if k is not a transmission eigenvalue (this will be part of a future work). We shall say that (13) is well-posed if it admits a unique solution for all $f \in H^{3/2}(\partial D), g \in H^{1/2}(\partial D)$.

Proposition 2. *Assume that $k > 0$ is not a transmission eigenvalue. Then the operator $G : R(\overline{H}) \rightarrow L^2(\mathbb{S}^{d-1})$ is compact, injective with dense range.*

Proof. First we show the injectivity of G . Let $V = (v, \partial_v^+ v) \in \overline{R(H)}$ such that $GV = 0$. Then from the Rellich lemma, the solution u_s of (10) is zero in $\mathbb{R}^d \setminus \overline{D}$. Therefore, if we define $u = v + u_s$, then the pair (u, v) satisfies the interior transmission problem (13) with $f = g = 0$. Since we assumed that $k > 0$ is not a transmission eigenvalue, we deduce that $v = 0$ and so $V = 0$.

Now we focus our attention on the denseness of the range of G . First we establish an identity of symmetry. Let $V_1 = (v_1, \partial_v^+ v_1)$, $V_2 = (v_2, \partial_v^+ v_2) \in \overline{R(H)}$. Denote w_1, w_2 the corresponding solutions to Problem (10). In particular we have

$$\Delta w_1 + k^2 n w_1 = k^2(1-n)v_1, \quad \Delta w_2 + k^2 n w_2 = k^2(1-n)v_2 \quad \text{in } \mathbb{R}^d \setminus \Gamma. \quad (14)$$

Multiplying the first equation by w_2 and the second by w_1 , integrating by parts the difference over B_R , the open ball of radius R centered at O , we obtain

$$\begin{aligned} k^2 \int_D (n-1)(v_1 w_2 - v_2 w_1) dx \\ = \int_{\partial B_R} (\partial_v w_1 w_2 - w_1 \partial_v w_2) ds + \int_{\Gamma} ([w_2] \partial_v^+ v_1 - [w_1] \partial_v^+ v_2) ds. \end{aligned}$$

Taking the limit as $R \rightarrow +\infty$ and using that $\lim_{R \rightarrow +\infty} \int_{\partial B_R} (\partial_v w_1 w_2 - w_1 \partial_v w_2) ds = 0$ (w_1 and w_2 satisfy the radiation condition), we find the identity

$$k^2 \int_D (n-1)v_1 w_2 dx + \int_{\Gamma} \partial_v^+ v_1 [w_2] ds = k^2 \int_D (n-1)v_2 w_1 dx + \int_{\Gamma} \partial_v^+ v_2 [w_1] ds. \quad (15)$$

Using (15), we deduce that for $\phi, g \in L^2(\mathbb{S}^{d-1})$, we have

$$\begin{aligned} & \langle G(H\phi), \bar{g} \rangle_{L^2(\mathbb{S}^{d-1})} \\ &= k^2 \int_D (n-1)(H\phi + u_s(\phi))Hg \, dx + \int_{\Gamma} [H\phi + u_s(\phi)] \partial_v^+(Hg) \, ds \\ &= k^2 \int_D (n-1)(Hg + u_s(g))H\phi \, dx + \int_{\Gamma} [Hg + u_s(g)] \partial_v^+(H\phi) \, ds \\ &= \langle G(Hg), \bar{\phi} \rangle_{L^2(\mathbb{S}^{d-1})}. \end{aligned}$$

Therefore if $\bar{g} \in R(G)^\perp$ then $G(Hg) = 0$. The injectivity of G and H imply that $g = 0$ which shows that G has dense range.

Finally, using again the estimate $\|\partial_v v\|_{L^2(\Gamma)} \leq C\|v\|_{L^2(D)}$ for all $v \in \mathcal{H}$, results of interior regularity and the definition of H (see (6)), one can check that $H : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(D) \times L^2(\Gamma)$ is compact. Since $G = H^*T$ and T is continuous, we deduce that $G : L^2(D) \times L^2(\Gamma) \rightarrow L^2(\mathbb{S}^{d-1})$ is compact. \square

Proposition 3. *For all $V = (v, \partial_v^+ v) \in \overline{R(H)}$, we have the energy identity*

$$\Im m(\langle TV, V \rangle_{L^2(D) \times L^2(\Gamma)}) = k^2 \int_D \Im m(n) |u_s + v|^2 \, dx + k \|GV\|_{L^2(\mathbb{S}^{d-1})}^2, \quad (16)$$

where u_s denotes the solution of (10). As a consequence if $\Im m(n) \geq 0$ a.e. in D and if k is not a transmission eigenvalue of (13), then T is injective.

Proof. Multiplying by \bar{u}_s the equation $\Delta u_s + k^2 u_s = -k^2(n-1)(u_s + v)$ and integrating by parts over the ball B_R , we obtain

$$\begin{aligned} -k^2 \int_D (n-1)(u_s + v) \bar{u}_s \, dx &= \\ - \int_{B_R} |\nabla u_s|^2 - k^2 |u_s|^2 \, dx + \int_{\partial B_R} \partial_v u_s \bar{u}_s \, ds - \int_{\Gamma} \partial_v^+ u_s [\bar{u}_s] \, ds. \end{aligned} \quad (17)$$

Using (17), then we find

$$\begin{aligned} \langle TV, V \rangle_{L^2(D) \times L^2(\Gamma)} &= k^2 \int_D (n-1) |u_s + v|^2 \, dx - \int_{B_R} |\nabla u_s|^2 - k^2 |u_s|^2 \, dx \\ &\quad + \int_{\Gamma} [v + u_s] \partial_v^+ \bar{v} \, ds - \int_{\Gamma} \partial_v^+ u_s [\bar{u}_s] \, ds + \int_{\partial B_R} \partial_v u_s \bar{u}_s \, ds. \end{aligned}$$

Since $\partial_v^+ u_s = -\partial_v^+ v$ and $[v] = 0$ (interior regularity) on Γ , we deduce

$$\begin{aligned} \langle TV, V \rangle_{L^2(D) \times L^2(\Gamma)} &= k^2 \int_D (n-1) |u_s + v|^2 \, dx - \int_{B_R} |\nabla u_s|^2 - k^2 |u_s|^2 \, dx \\ &\quad - 2\Re e \left(\int_{\Gamma} [u_s] \partial_v^+ \bar{u}_s \, ds \right) + \int_{\partial B_R} \partial_v u_s \bar{u}_s \, ds. \end{aligned} \quad (18)$$

The radiation condition (see (1)) implies $\lim_{R \rightarrow \infty} \int_{\partial B_R} \partial_v u_s \bar{u}_s \, ds = ik \int_{\mathbb{S}^{d-1}} |u_s^\infty|^2 d\theta = ik \|GV\|_{L^2(\mathbb{S}^{d-1})}^2$. As a consequence, taking the imaginary part of (18) and letting R

goes to infinity, we get identity (16). Now if $TV = 0$ and if $\Im m(n) \geq 0$ a.e. in D , then (16) gives $GV = 0$. Since G is injective when k is not a transmission eigenvalue of (13) (Proposition 2), we deduce that T is injective. \square

4 Generalized Linear Sampling Method and Differential Linear Sampling Method

For $z \in \mathbb{R}^d$, we denote by $\Phi(.,z)$ the outgoing fundamental solution of the homogeneous Helmholtz equation such that

$$\Phi(x,z) = \frac{i}{4} H_0^{(1)}(k|x-z|) \text{ if } d = 2 \quad \text{and} \quad \frac{e^{ik|x-z|}}{4\pi|x-z|} \text{ if } d = 3. \quad (19)$$

Here $H_0^{(1)}$ stands for the Hankel function of first kind of order zero. The far field of $\Phi(.,z)$ is $\phi_z(\hat{x}) = e^{-ikz \cdot \hat{x}}$. The GLSM uses the following theorem whose proof is classical [7].

Theorem 1. *Assume that the interior transmission problem (13) is well-posed. Then*

$$z \in D \quad \text{if and only if} \quad \phi_z \in R(G).$$

The particularity of the GLSM is to build an approximate solution ($Fg \simeq \phi_z$) to the far field equation by minimizing the functional $J^\alpha(\phi_z, .) : L^2(\mathbb{S}^{d-1}) \rightarrow \mathbb{R}$ defined by

$$J^\alpha(\phi_z, g) = \alpha \langle F^\# g, g \rangle_{L^2(\mathbb{S}^{d-1})} + \|Fg - \phi_z\|_{L^2(\mathbb{S}^{d-1})}^2, \quad \forall g \in L^2(\mathbb{S}^{d-1}), \quad (20)$$

where $F^\# := |\frac{1}{2}(F + F^*)| + |\frac{1}{2i}(F - F^*)|$.

Theorem 2 (GLSM). *Assume that the interior transmission problem (13) is well-posed, that the index n satisfies $[\Im m(n) \geq 0, \Re e(n-1) \geq n_* \text{ a.e. in } D]$ or $[\Im m(n) \geq 0, \Re e(1-n) \geq n_* \text{ a.e. in } D]$ for some constant $n_* > 0$. Let $g_z^\alpha \in L^2(\mathbb{S}^{d-1})$ be a minimizing sequence of $J^\alpha(\phi_z, .)$ such that*

$$J^\alpha(\phi_z, g_z^\alpha) \leq \inf_g J^\alpha(\phi_z, g) + p(\alpha), \quad (21)$$

where $\lim_{\alpha \rightarrow 0} \alpha^{-1} p(\alpha) = 0$. Then

- $z \in D \quad \text{if and only if} \quad \lim_{\alpha \rightarrow 0} \langle F^\# g_z^\alpha, g_z^\alpha \rangle_{L^2(\mathbb{S}^{d-1})} < +\infty$.
- If $z \in D$ then there exists $h \in \overline{R(H)}$ such that $\phi_z = Gh$ and Hg_z^α converges strongly to h as $\alpha \rightarrow 0$.

Thus the GLSM, justified by this theorem, offers a way to recover D , that is to identify the perturbation in the reference background. Note that the GLSM, contrary to the LSM, provides an exact characterization of D . However it does not give any information on the location of the crack Γ .

Proof. We establish this theorem by applying the abstract result of [7, Theorem 2.10]. The latter requires that the following properties hold.

- i) $F = GH = H^*TH$ is injective with dense range and G is compact.
- ii) F^\sharp factorizes as $F^\sharp = H^*T^\sharp H$ where T^\sharp satisfies the coercivity property

$$\exists \mu > 0, \forall V \in \overline{R(H)}, \quad |\langle T^\sharp V, V \rangle_{L^2(D) \times L^2(\Gamma)}| \geq \mu \|V\|_{L^2(D) \times L^2(\Gamma)}^2; \quad (22)$$

$$iii) \quad V \mapsto |\langle T^\sharp V, V \rangle_{L^2(D) \times L^2(\Gamma)}|^{1/2} \text{ is uniformly convex on } \overline{R(H)}.$$

Item i) is a consequence of Propositions 1, 2 and 3. Moreover, we deduce iii) from ii) and from the fact that $\langle F^\sharp g, g \rangle_{L^2(D) \times L^2(\Gamma)} = \|(F^\sharp)^{1/2} g\|_{L^2(\mathbb{S}^{d-1})}^2$ (see e.g. [7]). Therefore, it remains to show ii). To proceed, we use [7, Theorem 2.31] which guarantees that it is true if :

- T injective on $\overline{R(H)}$;
- $\Im m(\langle TV, V \rangle_{L^2(D) \times L^2(\Gamma)}) \geq 0$ for all $V \in \overline{R(H)}$;
- $\Re e(T)$ decomposes as $\Re e(T) = T_0 + C$ where T_0 satisfies (22) and where C is compact on $\overline{R(H)}$.

The first two items have been proved in Proposition 3. Let us focus our attention on the last one. By definition, we have $TV = (k^2(n-1)(v+u_s), [v+u_s])$. Set $\tilde{C}V = (k^2(n-1)u_s, [v+u_s] - \partial_v^+ v|_\Gamma)$. Using results of interior regularity, one can check that $C = \Re e(\tilde{C})$ is compact. Now, define $T_0 := \Re e(T) - C = (k^2\Re e(n-1)v, \partial_v^+ v|_\Gamma)$. Clearly one has $|\langle T_0 V, V \rangle_{L^2(D) \times L^2(\Gamma)}| \geq n_* \|V\|_{L^2(D) \times L^2(\Gamma)}^2$ when $\Re e(n-1) \geq n_*$. The case $\Re e(1-n) \geq n_*$ can be dealt in a similar way. \square

When one has only acces to a noisy version F^δ of F , then $F^{\sharp, \delta}$ might not have the required factorization and the cost function (21) must be regularized. For this aspect, we refer the reader to [2, Section 5.2].

We now give the theoretical foundation of the DLSM which will allow us to localize the position of the crack Γ . The DLSM relies on the comparison of the solutions of the following interior transmission problems (without and with cracks).

$$\mathcal{P}(D) \begin{cases} \Delta u_0 + k^2 n u_0 = 0 & \text{in } D \\ \Delta v_0 + k^2 v_0 = 0 & \text{in } D \\ u_0 - v_0 = \Phi_z & \text{on } \partial D \\ \partial_v u_0 - \partial_v v_0 = \partial_v \Phi_z & \text{on } \partial D, \end{cases} \quad \mathcal{P}_\Gamma(D) \begin{cases} \Delta u + k^2 n u = 0 & \text{in } D \\ \Delta v + k^2 v = 0 & \text{in } D \\ \partial_v^\pm u = 0 & \text{on } \Gamma \\ u - v = \Phi_z & \text{on } \partial D \\ \partial_v u - \partial_v v = \partial_v \Phi_z & \text{on } \partial D, \end{cases} \quad (23)$$

where $u_0, v_0, u, v \in L^2(D)$, $u_0 - v_0 \in H^2(D)$ and $u - v \in H^1(D \setminus \Gamma)$ is such that $\Delta(u - v) \in L^2(D \setminus \Gamma)$. We split the domain D into two kinds of connected components (see Fig. 2): The ones containing cracks are listed by $(D_\Gamma^j)_j$; others are listed by $(D_0^j)_j$. And we set $D_\Gamma := \bigcup_j D_\Gamma^j$ and $D_0 := \bigcup_j D_0^j$ so that $D = D_\Gamma \cup D_0$.

Theorem 3. *Assume that Γ is a part of the boundary of a domain Ω such that $\partial\Omega$ is analytic. Assume that n is analytic in D_Γ and does not vanish. Assume also that*

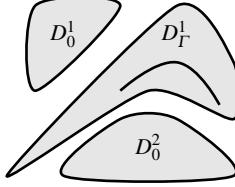


Fig. 2: We split D into two families of connected components.

k is not a Neumann eigenvalue for $-n^{-1}\Delta$ in Ω and is such that both $\mathcal{P}(D)$ and $\mathcal{P}_\Gamma(D)$ (see (23)) are well-posed.

i) If $z \in D_0$ then $v = v_0$ in D . ii) If $z \in D_\Gamma$ then $v \neq v_0$ in D_Γ .

Proof. i) Let $z \in D_0$. In D_0 , the equations for $\mathcal{P}(D)$ and $\mathcal{P}_\Gamma(D)$ coincide. By uniqueness of the solution for these problems, we deduce that $v = v_0$ in D_0 . On the other hand, one observes that $(0, -\Phi_z)$ satisfies the equations of $\mathcal{P}(D)$ and $\mathcal{P}_\Gamma(D)$ in D_Γ . As a consequence, by uniqueness of the solution for these problems, we also have $v = v_0 = -\Phi_z$ in D_Γ .

ii) Now let $z \in D_\Gamma$. We wish to show that $v \neq v_0$ in D_Γ . We proceed by contradiction assuming that $v = v_0$ in D_Γ . Define U such that $U = u - u_0$ in $D_\Gamma \setminus \Gamma$ and $U = 0$ in $\mathbb{R}^d \setminus D_\Gamma$. Since $U = \partial_v U = 0$ on ∂D_Γ , from the unique continuation principle, we find $U = 0$ in $\mathbb{R}^d \setminus \Gamma$ and so $\partial_v^\pm u_0 = 0$ on Γ (because $\partial_v^\pm u = 0$ on Γ). Furthermore the regularity of n implies that $\partial_v^\pm u_0$ is analytic on $\partial\Omega$ and we conclude that $\partial_v^\pm u_0 = 0$ on $\partial\Omega$. Since we assumed that k is not a Neumann eigenvalue for $-n^{-1}\Delta$ in Ω , we deduce that $u_0 = 0$ in Ω , and by unique continuation, $u_0 = 0$ in D_Γ . Thus we must have $v_0 = -\Phi_z$ in D_Γ which contradicts the fact that $u_0 - v_0 \in H^2(D)$. \square

Now we consider a first heterogeneous medium without crack with a perturbation of the reference background supported in \bar{D} modeled by some index n , and a second medium with the same n but with an additional crack inside D . The corresponding far field operators are denoted respectively F_0 and F_1 . Then for $j = 0, 1$, let $g_{j,z}^\alpha$ refer to the sequences introduced in the statement of Theorem 2 with $F_j^\sharp = |\frac{1}{2}(F_j + F_j^*)| + |\frac{1}{2i}(F_j - F_j^*)|$. We also set for $j = 0, 1$

$$\mathcal{A}_j^\alpha(z) = \langle F_j^\sharp g_{j,z}^\alpha, g_{j,z}^\alpha \rangle_{L^2(\mathbb{S}^{d-1})}; \quad \mathcal{D}_j^\alpha(z) = \langle F_j^\sharp (g_{1,z}^\alpha - g_{0,z}^\alpha), (g_{1,z}^\alpha - g_{0,z}^\alpha) \rangle_{L^2(\mathbb{S}^{d-1})}. \quad (24)$$

The combination of Theorem 2 and 3 leads to the following result.

Theorem 4 (DLSM). Assume that k , n and Γ are as in Theorem 3 and that n also satisfies the assumptions of Theorem 2. Then for $j = 0$ or 1

$$[z \in D_0] \Rightarrow [\lim_{\alpha \rightarrow 0} \mathcal{D}_j^\alpha(z) = 0] \quad \text{and} \quad [z \in D_\Gamma] \Rightarrow [0 < \lim_{\alpha \rightarrow 0} \mathcal{D}_j^\alpha(z) < +\infty].$$

Proof. As explained in the proof of Theorem 2, F_1^\sharp admits a factorization of the form $H^* T_1^\sharp H$ where T_1^\sharp is continuous and $\langle T_1^\sharp \cdot, \cdot \rangle$ is coercive. According to the study

of crack-free inhomogeneous medium a same factorization stands for F_0 involving an operator T_0^\sharp that have the same properties of T_1^\sharp . This implies (for $j = 0$ or 1) the existence of two positive constants κ and K such that

$$\kappa \|H(g_{1,z}^\alpha - g_{0,z}^\alpha)\|_{L^2(D)}^2 \leq \mathcal{D}_j^\alpha(z) \leq K \|H(g_{1,z}^\alpha - g_{0,z}^\alpha)\|_{L^2(D)}^2. \quad (25)$$

Now for $z \in D$, if we denote (u_0, v_0) (resp. (u_1, v_1)) the solution of $\mathcal{P}(D)$ (resp. $\mathcal{P}_\Gamma(D)$), then Theorem 2 and the GLSM for the crack-free inhomogeneous medium (see the justification in [7]) guarantee that $\lim_{\alpha \rightarrow 0} \|H(g_{1,z}^\alpha - g_{0,z}^\alpha)\| = \|H(v - v_0)\|$. Then the result follows from Theorem 3. \square

From Theorems 2 and 4, one can design indicators for D and D_Γ . Set for $j = 0$ or 1,

$$I^{\text{GLSM}}(z) = \lim_{\alpha \rightarrow 0} \frac{1}{\mathcal{A}_1^\alpha(z)} \quad \text{and} \quad I_j^{\text{DLSM}}(z) = \lim_{\alpha \rightarrow 0} \frac{1}{\mathcal{A}_0^\alpha(z) \left(1 + \frac{\mathcal{A}_0^\alpha(z)}{\mathcal{D}_j^\alpha(z)}\right)}. \quad (26)$$

For these indicators, one can show the following theorem which allows one to identify the connected components of D in which some cracks have appeared.

Corollary 1. *Under the assumptions of Theorem 4, we have for $j = 0$ or 1*

- $I^{\text{GLSM}}(z) = 0$ in $\mathbb{R}^d \setminus D$ and $I^{\text{GLSM}}(z) > 0$ in D .
- $I_j^{\text{DLSM}}(z) = 0$ in $\mathbb{R}^d \setminus D_\Gamma$ and $I_j^{\text{DLSM}}(z) > 0$ in D_Γ .

5 Numerical results

To conclude this work, we apply the GLSM and the DLSM on simulated backgrounds. All backgrounds have the same shape D constituted of three disjoint disks of radius 0.75 and of index $n = 1.5$. They differ from one to another in the distribution of cracks inside the disks. Admittedly, the straight cracks appearing in the backgrounds are not a portion of the boundary of an analytic domain. However, we expect that our algorithm remains robust when this theoretical assumption is not satisfied. For each background we generate a discretization of the far field operator F by solving numerically the direct problem for multiple incident fields $u_i(\theta_p)$ with wave number $k = 4\pi$. Then we compute the matrix $F = (u_s^\infty(\theta_p, \hat{x}_q))_{p,q}$ for θ_p, \hat{x}_q in $\{\cos(\frac{2l\pi}{100}), \sin(\frac{2l\pi}{100}), l = 1..100\}$ (somehow we discretize $L^2(\mathbb{S}^1)$). Finally, we add random noise to the simulated F and obtain our final synthetic far field data F^δ with $F_{pq}^\delta = F_{pq}(1 + \sigma N)$. Here N is a complex random variable whose real and imaginary parts are uniformly chosen in $[-1, 1]^2$. The parameter $\sigma > 0$ is chosen so that $\|F^\delta - F\| = 0.05\|F^\delta\|$.

5.1 GLSM

To handle the noise δ added on the far field data, we use a regularized version of the GLSM consisting in finding the minimizers $g_z^{\alpha, \delta}$ of the functional

$$g \mapsto J^{\alpha, \delta}(g) = \alpha(|\langle F^\# g, g \rangle_{L^2(\mathbb{S}^{d-1})}| + \delta \|F^\delta\| \|g\|_{L^2(\mathbb{S}^2)}^2) + \|F^\delta g - \phi_z\|_{L^2(\mathbb{S}^{d-1})}^2,$$

where $F^\# \delta := |\frac{1}{2}(F^\delta + F^{\delta*})| + |\frac{1}{2i}(F^\delta - F^{\delta*})|$. We fit α to δ according to [2, Section 5.2]. The new relevant indicator function for the regularized GLSM is then given by

$$I_{\text{GLSM}}^{\alpha, \delta}(z) = \frac{1}{\mathcal{A}^{\alpha, \delta}(z)}$$

where $\mathcal{A}^{\alpha, \delta}(z) = \langle F^\# g_z^{\alpha, \delta}, g_z^{\alpha, \delta} \rangle_{L^2(\mathbb{S}^{d-1})} + \delta \|F^\delta\| \|g_z^{\alpha, \delta}\|_{L^2(\mathbb{S}^{d-1})}^2$.

Fig. 3 shows the results of GLSM indicator function $z \mapsto I_{\text{GLSM}}^{\alpha, \delta}(z)$ for two different configurations where the second one is obtained from the first one by adding a crack to the third component. The two other components contain the same crack. One observes that GLSM is capable of retrieving the domain D for each configuration. We also observe how the behavior of the indicator function is different inside the third component. This is somehow what the DLSM exploits to isolate the component where a defect appears and this is what is discussed next.

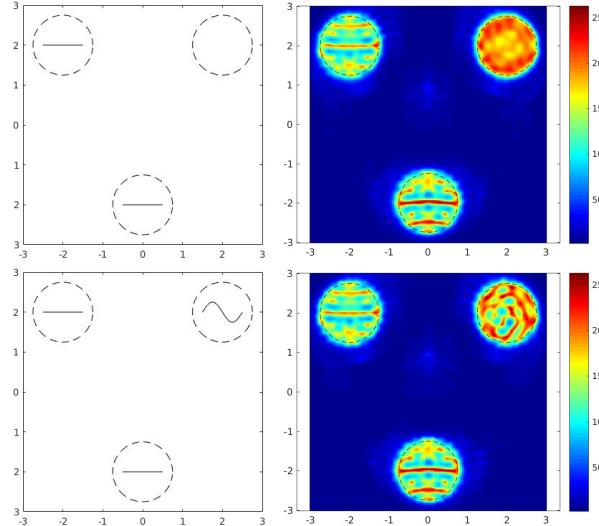


Fig. 3: Simulated backgrounds on the left and associated GLSM indicator function $z \mapsto I_{\text{GLSM}}^{\alpha, \delta}(z)$ on the right.

5.2 DLSM

Given two far field data F_0^δ and F_1^δ , we respectively define $F_0^{\sharp\delta}$, $g_{0,z}^{\alpha,\delta}$, $\mathcal{A}_0^{\alpha,\delta}(z)$ and $F_1^{\sharp\delta}$, $g_{1,z}^{\alpha,\delta}$, $\mathcal{A}_1^{\alpha,\delta}(z)$ associated to each data as described in the previous paragraph. We also define

$$\mathcal{D}^{\alpha,\delta}(z) = \langle F_0^{\sharp}(g_{1,z}^{\alpha,\delta} - g_{0,z}^{\alpha,\delta}), (g_{1,z}^{\alpha,\delta} - g_{0,z}^{\alpha,\delta}) \rangle_{L^2(\mathbb{S}^{d-1})}.$$

Then, according to (26), the DLSM indicator is given by

$$I_{\text{DLSM}}^{\alpha,\delta}(z) = \frac{1}{\mathcal{A}_0^{\alpha,\delta}(z) \left(1 + \frac{\mathcal{A}_0^{\alpha,\delta}(z)}{\mathcal{D}^{\alpha,\delta}(z)} \right)}.$$

The behavior of the DLSM indicator function is illustrated below for several scenarios shown in Fig. 4-7. In each figure is presented from left to right, the initial background (associated with F_0^δ), the damaged background (associated with F_1^δ) and the DLSM indicator function $z \mapsto I_{\text{DLSM}}^{\alpha,\delta}(z)$. As expected, the latter allows us to identify for all scenarios the component(s) D_Γ where (additional) cracks appeared. We also remark that it slightly accentuates the border of D_0 . But this effect is not explained by our theory and it does not contradict it: Our theoretical result does not stipulate that the indicator function is “uniformly” close to 0 outside D_Γ .

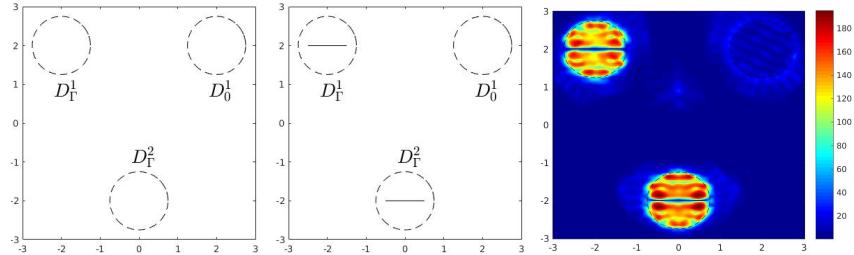


Fig. 4: A scenario for DLSM simulating the emergence of cracks in two components of a defect free background.

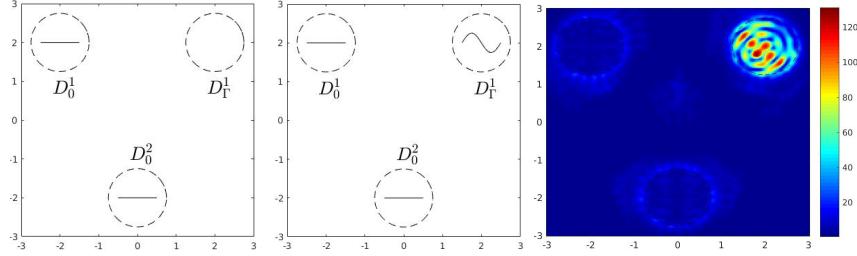


Fig. 5: A scenario for DLSM simulating the emergence of a crack in a healthy component of an already damaged background.

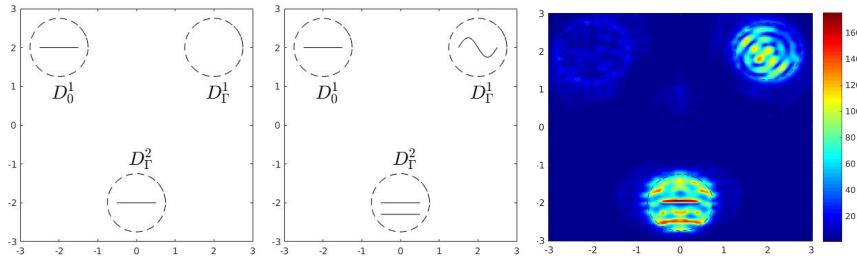


Fig. 6: A scenario for DLSM simulating the emergence of additional cracks in a healthy and a damaged components of an already damaged background.

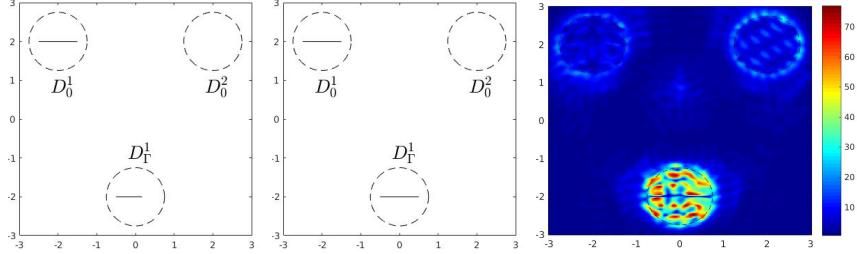


Fig. 7: A scenario for DLSM simulating the increase of the crack size in one component of an already damaged background.

6 Conclusion

We analyzed the DLSM to identify emergence of cracks embedded in an unknown background and image defective components from differential measurements of far

field data at a fixed frequency. The analysis is based on the justification of the GLSM for backgrounds with cracks which necessitates the study of a special interior transmission problem and the derivation of specific factorizations of the far field operator. The numerical tests on toy problems show that our method is reliable for different scenarios simulating the appearance of cracks between two measurements campaigns. This is a first step before addressing practical problems where the issues of limited aperture data and/or highly cluttered backgrounds should be solved.

References

1. L. Audibert, *Qualitative methods for heterogeneous media*. PhD thesis, École Doctorale Polytechnique, 2015.
2. L. Audibert, H. Haddar, *A generalized formulation of the linear sampling method with exact characterization of targets in terms of farfield measurements*. Inverse Problems, 30 (035011), 2014.
3. L. Audibert, A. Girard, H. Haddar, *Identifying defects in an unknown background using differential measurements*. Inverse Problems & Imaging, 9 (3) : 625-643, 2015.
4. F. Ben Hassen , Y. Boukari, H. Haddar, *Application of the linear sampling method to retrieve cracks with impedance boundary conditions*, Rapport de recherche RR-7478, INRIA, 2010.
5. L. Bourgeois, E. Lunéville, *On the use of the Linear Sampling Method to identify cracks in elastic waveguides*. Inverse Problems, 29 025017, 2013.
6. F. Cakoni, D. Colton, *Qualitative methods in inverse scattering theory Interaction of Mechanics and Mathematics: An Introduction*, Springer, 2006 .
7. F. Cakoni, D. Colton, H. Haddar, *Inverse Scattering Theory and Transmission Eigenvalues*. Series in Applied Mathematics, 2016.
8. D. Colton, R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*. Springer-Verlag, 1992.
9. D. Colton, R. Kress, *Integral Equation Methods in Scattering Theory*. John Wiley, 1993.
10. A. Kirsch, *An Introduction to the Mathematical Theory of Inverse Problems*. Springer, 2011.
11. A. Kirsch, S. Ritter *A linear sampling method for inverse scattering from an open arc*, Inverse Problems 16 89, 2000.
12. W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, 2000.